

TRIANGULATION

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CONTENTS

1	Introduction	2
2	Triangulation by Points	4
3	Triangulation by Sides	6
4	Linear System	7
5	Nine-point center	8
6	Trilinear Coordinates	15
7	Plane in Space	18

LIST OF FIGURES

Figure 1	Point P at a given distances from the vertices of the triangle.	2
Figure 2	Oriented area of the triangle	3
Figure 3	Nine-point circle	8
Figure 4	Triangle $A(0,0)$, $B(m,0)$, $C(n,p)$ and points T, H, F . .	13
Figure 5	Sine Rule	16
Figure 6	Normal $\vec{n}$ to the plane α and the point $P \in \alpha$	19

LIST OF TABLES

Table 1	Trilinears	15
Table 2	Exact Trilinears	17

ABSTRACT

The text links the three meanings of the word triangulation. The first is to find the coordinates of the point by the distance to the three given points. The second is the same by distances to the sides of the given triangle. Third, triangulation is a division a figure into triangles. We also considered coordinates of nine-point center, and trilinear coordinates.

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1 INTRODUCTION

When you know the coordinates of the points $A(A_x, A_y)$, $B(B_x, B_y)$, $C(C_x, C_y)$, $P(\xi, \eta)$ then we can calculate the distances:

$$\begin{cases} R_A = \sqrt{(\xi - A_x)^2 + (\eta - A_y)^2}, \\ R_B = \sqrt{(\xi - B_x)^2 + (\eta - B_y)^2}, \\ R_C = \sqrt{(\xi - C_x)^2 + (\eta - C_y)^2}. \end{cases} \quad (1)$$

Reverse case is more complicated.

In the following, we will keep these labels, using the standard mark for triangles in geometry, as in Figure 1.

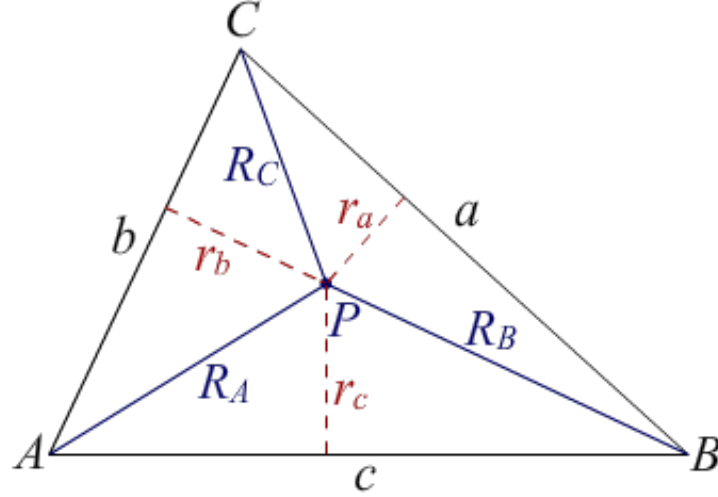


Figure 1: Point P at a given distances from the vertices of the triangle.

The Figure 1 shows the area of the triangle ABC has three areas:

$$\mu(ABC) = \mu(PAB) + \mu(PBC) + \mu(PCA). \quad (2)$$

It writes short $\mu = \mu_a + \mu_b + \mu_c$ where oriented areas are:

$$\begin{cases} \text{BCP: } \mu_a = \pm ar_a/2, \\ \text{CAP: } \mu_b = \pm br_b/2, \\ \text{ABP: } \mu_c = \pm cr_c/2, \end{cases} \quad (3)$$

where the area of the triangle is positive (negative) if the specified vertex circuit in a positive (negative) direction. The positive direction of circuiting points is reversed clockwise.

In the rectangular Cartesian coordinate system Oxy , the use of labels $A(A_x, A_y)$, $B(B_x, B_y)$, $C(C_x, C_y)$ gives the area (2):

$$\mu(ABC) = \frac{1}{2} \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}. \quad (4)$$

Why? Note that the area of the triangle ABC in Figure 2 consists of area trapezoid $A_x C_x CA$ plus $C_x B_x BC$ minus $A_x B_x BA$. We know that the area of a trapezoid is equal to the product half of the sum of bases (midline trapezoid) and height. In other words, we have $\mu(ABC) =$

$$= \frac{C_y + A_y}{2}(C_x - A_x) + \frac{B_y + C_y}{2}(B_x - C_x) - \frac{B_y + A_y}{2}(B_x - A_x)$$

$$\begin{aligned}
&= \frac{1}{2} [A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)] \\
&= \frac{1}{2} \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix},
\end{aligned}$$

and that is (4).

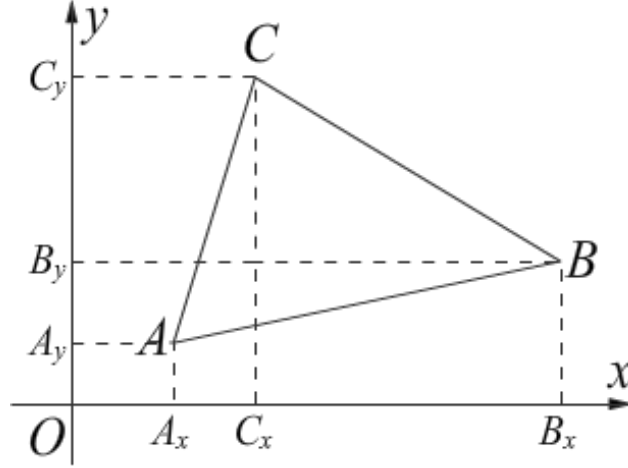


Figure 2: Oriented area of the triangle

The formula (4) can be written:

$$2\mu(ABC) = [A, B] + [B, C] + [C, A], \quad (5)$$

with the 'commutators':

$$[A, B] = A_x B_y - B_x A_y, \quad [B, C] = B_x C_y - C_x B_y, \quad [C, A] = C_x A_y - A_x C_y.$$

The same result (4) is equal to, successive:

$$\begin{aligned}
\mu(ABC) &= \frac{1}{2} [A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)], \\
\mu(ABC) &= \frac{1}{2} (A_x a_y + B_x b_y + C_x c_y),
\end{aligned} \quad (6)$$

where are oriented projections of sides

$$a_y = B_y - C_y, \quad b_y = C_y - A_y, \quad c_y = A_y - B_y.$$

Note that applies symmetrically:

$$A_x a_y + B_x b_y + C_x c_y = A_y a_x + B_y b_x + C_y c_x.$$

Furthermore, applying the commutators on projections, we get:

$$2\mu(ABC) = [a, b] = [b, c] = [c, a]. \quad (7)$$

Namely,

$$\begin{aligned}
[a, b] &= a_x b_y - b_x a_y = \\
&= (C_x - B_x)(A_y - C_y) - (A_x - C_x)(C_y - B_y) \\
&= (A_x B_y - B_x A_y) + (B_x C_y - C_x B_y) + (C_x A_y - A_x C_y)
\end{aligned}$$

$$= [A, B] + [B, C] + [C, A] = 2\mu(ABC),$$

and so on.

For example, let us prove:

$$a_x[C, A] - [B, C]b_x = 2C_x\mu(ABC). \quad (8)$$

Indeed, we have the sequence:

$$\begin{aligned} & a_x[C, A] - [B, C]b_x = \\ &= (C_x - B_x)(C_x A_y - A_x C_y) - (B_x C_y - C_x B_y)(A_x - C_x) \\ &= (C_x^2 A_y - A_x C_x C_y - B_x C_x A_y + \underline{A_x B_x C_y}) \\ &\quad - (\underline{A_x B_x C_y} - B_x C_x C_y - A_x C_x B_y + C_x^2 B_y) \\ &= C_x[(A_x B_y - B_x A_y) + (B_x C_y - C_x B_y) + (C_x A_y - A_x C_y)] \\ &= C_x([A, B] + [B, C] + [C, A]) = C_x \cdot 2\mu. \end{aligned}$$

2 TRIANGULATION BY POINTS

Theorem 1. *If the vertices of the triangle are $A(A_x, A_y)$, $B(B_x, B_y)$, $C(C_x, C_y)$ at distances R_A, R_B, R_C from the point $P(\xi, \eta)$, then the coordinates of the point:*

$$\begin{cases} \xi = +\frac{1}{4\mu}[(R_A^2 - A^2)a_y + (R_B^2 - B^2)b_y + (R_C^2 - C^2)c_y], \\ \eta = -\frac{1}{4\mu}[(R_A^2 - A^2)a_x + (R_B^2 - B^2)b_x + (R_C^2 - C^2)c_x], \end{cases} \quad (9)$$

where $Q^2 = Q_x^2 + Q_y^2$, $Q \in \{A, B, C\}$, and μ is the area of the given triangle.

Proof. From $\overline{PA} = R_A$, $\overline{PB} = R_B$ and $\overline{PC} = R_C$, Figure 1, follows

$$\begin{cases} (A_x - \xi)^2 + (A_y - \eta)^2 = R_A^2, \\ (B_x - \xi)^2 + (B_y - \eta)^2 = R_B^2, \\ (C_x - \xi)^2 + (C_y - \eta)^2 = R_C^2. \end{cases}$$

Subtracting the first from the second and third equations, we get:

$$\begin{cases} (B_x - A_x)\xi + (B_y - A_y)\eta = \frac{1}{2}(B_x^2 - A_x^2 + B_y^2 - A_y^2 - R_B^2 + R_A^2) \\ (C_x - B_x)\xi + (C_y - B_y)\eta = \frac{1}{2}(C_x^2 - B_x^2 + C_y^2 - B_y^2 - R_C^2 + R_B^2), \end{cases}$$

i.e.

$$\begin{cases} c_x \xi + c_y \eta = \frac{1}{2}[(B^2 - A^2) - (R_B^2 - R_A^2)] \\ a_x \xi + a_y \eta = \frac{1}{2}[(C^2 - B^2) - (R_C^2 - R_B^2)], \end{cases} \quad (10)$$

where

$$a_x = C_x - B_x, \quad a_y = C_y - B_y, \quad c_x = B_x - A_x, \quad c_y = B_y - A_y$$

are oriented projections of the sides a, b, c .

It is easy to obtain the determinants of this system:

$$\Delta = \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} = 2\mu, \quad (11)$$

$$\Delta_x = \frac{1}{2} \begin{vmatrix} A^2 - R_A^2 & B^2 - R_B^2 & C^2 - R_C^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}, \quad (12)$$

$$\Delta_y = -\frac{1}{2} \begin{vmatrix} A^2 - R_A^2 & B^2 - R_B^2 & C^2 - R_C^2 \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix}, \quad (13)$$

Accordingly, the coordinates of the point $P(\xi, \eta)$ are $\xi = \frac{\Delta_x}{\Delta}$ and $\eta = \frac{\Delta_y}{\Delta}$ which is supposed to prove. $\square$

Example 1. Apply Theorem 1 to find the coordinates of circumcenter $P_o(x_o, y_o)$ of the triangle.

Solution. When in the theorem 1 we have $R_A = R_B = R_C = R$, that is radius of circumcenter of a triangle, then the determinant of x , we can write

$$\begin{aligned} \Delta_x &= \frac{1}{2} \begin{vmatrix} A^2 - R_A^2 & B^2 - R_B^2 & C^2 - R_C^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} = \\ &= \frac{1}{2} \begin{vmatrix} A^2 & B^2 & C^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} R^2 & R^2 & R^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}. \end{aligned}$$

The second determinant is zero, because the first and third rows are proportional. Similarly we obtain for the determinant of y . Therefore, triangulation with equal distances to the given points is reduced to the search center inscribed circle of triangle $P(x_o, y_o)$ with coordinates:

$$\begin{cases} x_o : & -\frac{1}{4\mu}(A^2 a_y + B^2 b_y + C^2 c_y), \\ y_o : & +\frac{1}{4\mu}(A^2 a_x + B^2 b_x + C^2 c_x). \end{cases} \quad (14)$$

This is what we should find. $\square$

Note, this is the same result as [1]: Theorem 2.

Example 2. Given points $A(5, 1)$, $B(-1, 4)$, $C(1, -1)$ and $P(3, 4)$. Find the distances R and check the Theorem 1.

Solution. Squares of the distances to point P from the vertices of the given triangle are:

$$R_A^2 = 13, \quad R_B^2 = 16, \quad R_C^2 = 29,$$

and the differences of the squares:

$$R_A^2 - A^2 = -13, \quad R_B^2 - B^2 = -1, \quad R_C^2 - C^2 = 27.$$

Next, we check the formula (2). Oriented projections of the sides are:

$$a_x = C_x - B_x = 2, \quad a_y = C_y - B_y = -5,$$

$$b_x = A_x - C_x = 4, \quad b_y = A_y - C_y = 2,$$

$$c_x = B_x - A_x = -6, \quad c_y = B_y - A_y = 3.$$

Hence the determinants:

$$\begin{cases} \Delta : & 2\mu = [a, b] = a_x b_y - b_x a_y = 2 \cdot 2 - 4 \cdot (-5) = 24, \\ \Delta_x : & \frac{1}{2}[-13 \cdot (-5) - 1 \cdot 2 + 27 \cdot 3] = 72, \\ \Delta_y : & -\frac{1}{2}[-13 \cdot 2 - 1 \cdot 4 + 27 \cdot (-6)] = 96. \end{cases} \quad (15)$$

Then easily find quotients $P(P_x, P_y)$:

$$P_x = \frac{\Delta_x}{\Delta} = 3, \quad P_y = \frac{\Delta_y}{\Delta} = 4,$$

as was supposed to get. $\square$

After these examples we can understand why the point P is ‘predefined’ by its distances to the vertices of the given triangle. However, the redundant data is somewhat useful in practice. When you stand on the Earth’s surface at the point P and measure three distances to three known positions (A, B, C) in the vicinity, then triangulation calculate our position, and we find if we have some measurement error.

3 TRIANGULATION BY SIDES

The equation of a straight line through two given points A, B :

$$y - A_y = \frac{B_y - A_y}{B_x - A_x}(x - A_x).$$

This is a straight line on which is the side c . It can be written like this:

$$c : c_y x - c_x y = [A, B]. \quad (16)$$

The distance from the point $P(x_0, y_0)$ to the line (c) :

$$d(P, c) = \frac{c_y x - c_x y - [A, B]}{-\sqrt{c_x^2 + c_y^2}},$$

or

$$-cr_c = c_y x - c_x y - [A, B], \quad (17)$$

which is in accordance with the triangle in Figure 1.

Theorem 2. *Given triangle ABC . If the point $P(P_x, P_y)$ is at the distances r_a, r_b and r_c from the sides of the triangle, then its coordinates are:*

$$P_x = \frac{\mu_a A_x + \mu_b B_x + \mu_c C_x}{\mu_a + \mu_b + \mu_c}, \quad P_y = \frac{\mu_a A_y + \mu_b B_y + \mu_c C_y}{\mu_a + \mu_b + \mu_c}. \quad (18)$$

Proof. Starting from the general form of the equation (15) and the expression (16) for a distance of points $P(\xi, \eta)$ from side $a = BC$, $b = CA$ and $c = AB$ of the triangle ABC we arrive to the system of equations:

$$\begin{cases} a_y \xi - a_x \eta - [B, C] &= -ar_a, \\ b_y \xi - b_x \eta - [C, A] &= -br_b, \\ c_y \xi - c_x \eta - [A, B] &= -cr_c. \end{cases}$$

Adding all three equations we obtain the identity $-2\mu = -2\mu$, which means that this system is consistent, and that the choice of the sign on the right side of the equations are good. Any two of the three equations are sufficient for a solution. From the first two we find the determinants:

$$\Delta = 2\mu, \quad \Delta_\xi = 2\mu C_x - a_x br_b + b_x ar_a, \quad \Delta_\eta = 2\mu C_y - a_y br_b + b_y ar_a.$$

Then, we have $\Delta = 2\mu$, and using (8):

$$\Delta_\xi = 2\mu_a A_x + 2\mu_b B_x + 2\mu_c C_x, \quad \Delta_\eta = 2\mu_a A_y + 2\mu_b B_y + 2\mu_c C_y.$$

Dividing the determinants obtain coordinates of the point $P(\xi, \eta)$:

$$\xi = \frac{\mu_a A_x + \mu_b B_x + \mu_c C_x}{\mu_a + \mu_b + \mu_c}, \quad \eta = \frac{\mu_a A_y + \mu_b B_y + \mu_c C_y}{\mu_a + \mu_b + \mu_c},$$

as it is supposed to prove. $\square$

Example 3. Apply the Theorem 2 to find the center $P_1(x_1, y_1)$ of incircle, i.e. inscribed circle of a given triangle.

Solution. When is given the center of inscribed circle, incenter P_1 , then $r_a = r_b = r_c = r$, that is the radius of the incircle, by (2) we have:

$$\mu_a = \frac{ar}{2}, \mu_b = \frac{br}{2}, \mu_c = \frac{cr}{2}, \quad \mu = \frac{r(a+b+c)}{2}. \quad (19)$$

Hence (3) we get:

$$x_1 = \frac{aA_x + bB_x + cC_x}{a+b+c}, \quad y_1 = \frac{aA_y + bB_y + cC_y}{a+b+c}. \quad (20)$$

These are the required formulas of the incircle. $\square$

However, when the point $P_c(x_c, y_c)$ is the center of outside inscribed circle, i.e. excircle on the side c of the triangle, then the area $\mu(ABP_c) = \mu_c$ is negative and expression (19) becomes:

$$x_c = \frac{aA_x + bB_x - cC_x}{a+b-c}, \quad y_c = \frac{aA_y + bB_y - cC_y}{a+b-c}. \quad (21)$$

These expressions (20-21) are known in analytical geometry.

4 LINEAR SYSTEM

Note that the coordinates of the center of the circle, $P_o(\xi, \eta)$ in example 1, we can get by solving systems of linear equations:

$$\begin{cases} A_x \xi + A_y \eta + \zeta = \frac{A_x^2 + A_y^2}{2}, \\ B_x \xi + B_y \eta + \zeta = \frac{B_x^2 + B_y^2}{2}, \\ C_x \xi + C_y \eta + \zeta = \frac{C_x^2 + C_y^2}{2}. \end{cases} \quad (22)$$

Orthocenter for $\xi = H_x$ and $\eta = H_y$ is solution of the system of linear equations:

$$\begin{cases} A_x \xi + A_y \eta + \zeta = -B_x C_x - B_y C_y, \\ B_x \xi + B_y \eta + \zeta = -C_x A_x - C_y A_y, \\ C_x \xi + C_y \eta + \zeta = -A_x B_x - A_y B_y. \end{cases} \quad (23)$$

Appropriate system for $P(\xi, \eta)$ with the Triangulation by Points:

$$\begin{cases} A_x \xi + A_y \eta + \zeta = A^2 - R_A^2, \\ B_x \xi + B_y \eta + \zeta = B^2 - R_B^2, \\ C_x \xi + C_y \eta + \zeta = C^2 - R_C^2. \end{cases} \quad (24)$$

In general, the solution of the system:

$$\begin{cases} A_x \xi + A_y \eta + \zeta = E_A, \\ B_x \xi + B_y \eta + \zeta = E_B, \\ C_x \xi + C_y \eta + \zeta = E_C, \end{cases} \quad (25)$$

is the point $P(\xi, \eta)$, where E_A, E_B, E_C are arbitrary real numbers. The same system in matrix notation

$$\hat{M}\vec{X} = \vec{E}, \quad (26)$$

where:

$$\hat{M} = \begin{pmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{pmatrix}, \quad \vec{X} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \vec{E} = \begin{pmatrix} E_A \\ E_B \\ E_C \end{pmatrix}.$$

If a straight line contains the points $\vec{P}_1, \vec{P}_2$, then for every real number t it contains a point $\vec{P} = \vec{P}_1 + (\vec{P}_2 - \vec{P}_1)t$. Multiply this vector equation to the left by matrix $\hat{M}$ and obtain $\vec{E} = \vec{E}_1 + (\vec{E}_2 - \vec{E}_1)t$. The conclusion is, 'collinear' $\vec{P}$ has been generated by 'collinear' $\vec{E}$ and vice versa.

Theorem 3. Given a triangle ABC of a nonzero area. For each set of numbers $\vec{E}$ there is a unique sequence of numbers $\vec{X}$ such that matrix equality (26) is true and vice versa, for every $\vec{X}$ there is a unique $\vec{E}$.

Proof. When is given a series of $\vec{E}$, then the system has a unique solution $\vec{X}$ because a matrix $\hat{M}$ is regular. The $\det \hat{M} = 2\mu(ABC) \neq 0$. Conversely, when is given a series of $\vec{X}$, then there is a unique set of $\vec{E} = \hat{M}\vec{X}$. $\square$

However, to find these sequences $\vec{X}$ and $\vec{E}$, sometimes is difficult job.

5 NINE-POINT CENTER

According Kimberling's classification, significant points of the triangle are the Incenter (X_1 , here P_1), Centroid (X_2 , here G), Circumcenter (X_3 , here P_0 or F), Orthocenter (X_4 , here H), then the Nine-point center (X_5 , here S_9). This fifth point is the center S_9 of the circle containing nine points of the triangle, which we will mark by letters A, B, C with indexes 1, 2, 3.

The *nine-point* of the triangle are (as opposed to the vertices the same name): three middle of sides (A_1, B_1, C_1), three feet of the altitudes (A_2, B_2, C_2) and three points (A_3, B_3, C_3) which are the middle points from vertices to the orthocenter. The following theorem is also known.

Theorem 4. Each of the nine points A_i, B_i, C_i , $i \in \{1, 2, 3\}$ lie on the same circle with center S_9 .

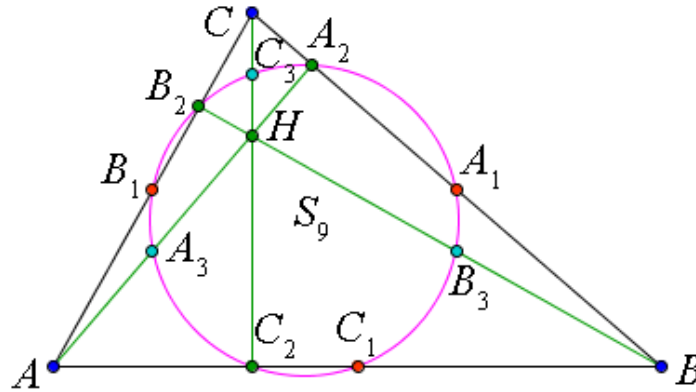


Figure 3: Nine-point circle

Proof. In the triangle ABC at Figure 3, because the C_1 and B_1 are mid-points of AB and AC, must be $C_1B_1 \parallel BC$. That is because the middle line of the triangle is parallel to the base. Also, in the triangle BCH, because B_3 and

C_3 are middle points of sides BH and CH, implies $B_3C_3 \parallel BC$. Es $B_1C_1 \parallel BC$ and $B_3C_3 \parallel BC$ it is $B_1C_1 \parallel B_3C_3$.

In triangle BAH, because B_3 and C_1 are middle-points of sides BH and BA, we have $B_3C_1 \parallel HA$. Similarly, in the triangle CAH, because B_1 and C_3 are middle points of AC and HC, must be $B_1C_3 \parallel HA$. From $B_3C_1 \parallel HA$ and $B_1C_3 \parallel HA$ follows $B_3C_1 \parallel B_1C_3$. Also, apply the implications:

$$B_3C_1 \parallel HA \wedge HA \subset AA_2 \Rightarrow B_3C_1 \parallel AA_2,$$

$$AA_2 \perp BC \wedge BC \parallel B_1C_1 \Rightarrow AA_2 \perp B_1C_1,$$

$$B_3C_1 \parallel AA_2 \wedge AA_2 \perp B_1C_1 \Rightarrow B_3C_1 \perp B_1C_1.$$

Therefore, quadrilateral $C_1B_1C_3B_3$ is a rectangle. As the opposite angles of such quadrilateral are supplementary, it follows that it can be inscribed into the circle. Similarly, quadrilateral $C_1A_3C_3A_1$ is a rectangle, so that it can be enrolled into a circle. Thus, the points $C_1, A_3, B_1, C_3, A_1, B_3$ are on the same circle with a diameter C_1C_3 , because this is diagonal of the both rectangles. Center of the circle is the midpoint of C_1C_3 . That is the point S_9 .

As AA_2 is height of triangle, angle $A_3A_2A_1$ is right angle. However, line A_3A_1 is also the diameter of the circle (it is the diagonal of a rectangle $C_1A_3C_3A_1$), and the point A_2 has to be on the same circle. Similarly, the point C_2 and B_2 are on the circle. Thus, all nine points A_i, B_i, C_i , where $i \in \{1, 2, 3\}$ belong to the same circle, which is called the nine-point circle. $\square$

To construct this center S_9 , first find the middle-points A_1, B_1, C_1 that are versus the same named vertexes. This gives the midlines to form the triangle $A_1B_1C_1$ inscribed into the given triangle ABC. Then construct the bisectors of sides of the triangle $A_1B_1C_1$, which intersects at the point S_9 . Point S_9 is the center of the circle that passes, inter alia, the points A_1, B_1, C_1 .

Example 4. Find the coordinates of nine-point center.

Solution. Center of the nine-point of the triangle ABC is the circumcenter of the triangle $A_1B_1C_1$. Therefore, we can apply the result of the example 1, so we are looking for the circumcenter of the triangle with vertices:

$$\begin{cases} a: & A_1 \left(\frac{C_x+B_x}{2}, \frac{C_y+B_y}{2} \right) \\ b: & B_1 \left(\frac{A_x+C_x}{2}, \frac{A_y+C_y}{2} \right) \\ c: & C_1 \left(\frac{B_x+A_x}{2}, \frac{B_y+A_y}{2} \right), \end{cases} \quad (27)$$

and (14). These mid-points define triangle $A_1B_1C_1$ whose circumcenter is the center of the nine-point triangle ABC.

When a given triangle ABC, we find a point S_9 as circumcenter of triangle $A_1B_1C_1$. In the denominator of (11) then we have the determinant:

$$\begin{aligned} \Delta(S_9) &= 2\mu(A_1B_1C_1) = \begin{vmatrix} A_{1x} & B_{1x} & C_{1x} \\ A_{1y} & B_{1y} & C_{1y} \\ 1 & 1 & 1 \end{vmatrix} = \\ &= \begin{vmatrix} \frac{B_x+C_x}{2} & \frac{C_x+A_x}{2} & \frac{A_x+B_x}{2} \\ \frac{B_y+C_y}{2} & \frac{C_y+A_y}{2} & \frac{A_y+B_y}{2} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{4} \begin{vmatrix} B_x+C_x & C_x+A_x & A_x+B_x \\ B_y+C_y & C_y+A_y & A_y+B_y \\ 1 & 1 & 1 \end{vmatrix} = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left(\begin{vmatrix} B_x & C_x & A_x \\ B_y + C_y & C_y + A_y & A_y + B_y \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} C_x & A_x & B_x \\ B_y + C_y & C_y + A_y & A_y + B_y \\ 1 & 1 & 1 \end{vmatrix} \right) \\
&= \frac{1}{4} \left(\begin{vmatrix} A_x & B_x & C_x \\ A_y + B_y & B_y + C_y & C_y + A_y \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} A_x & B_x & C_x \\ C_y + A_y & A_y + B_y & B_y + C_y \\ 1 & 1 & 1 \end{vmatrix} \right) \\
&= \frac{1}{4} \begin{vmatrix} A_x & B_x & C_x \\ A_y + 3G_y & B_y + 3G_y & C_y + 3G_y \\ 1 & 1 & 1 \end{vmatrix},
\end{aligned}$$

where $G_y = (A_y + B_y + C_y)/3$, that is ordinate of the center of gravity (centroid). This determinant at the middle row can be split into two, of which the second addend is determinant whose value is zero, because its second and third rows are proportional. Thus we get:

$$\Delta(S_9) = \frac{1}{4} \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \mu(ABC). \quad (28)$$

This is the same determinant expressed by coordinates of the triangle vertices ABC but its value is half the area of a triangle ABC . In other words, the area of the triangle $A_1B_1C_1$ equals to the fourth of the triangle ABC .

Then we find the determinants of variables, using the abbreviations $Q_1^2 = Q_{1x}^2 + Q_{1y}^2$, for $Q_1 \in \{A_1, B_1, C_1\}$. Thus, the determinant variable x is:

$$\begin{aligned}
\Delta_x(S_9) &= \frac{1}{2} \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_{1y} & B_{1y} & C_{1y} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{4} \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ B_y + C_y & C_y + A_y & A_y + B_y \\ 1 & 1 & 1 \end{vmatrix} = \\
&= \frac{1}{4} \left(\begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ B_y & C_y & A_y \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ C_y & A_y & B_y \\ 1 & 1 & 1 \end{vmatrix} \right) \\
&= \frac{1}{4} \left(\begin{vmatrix} C_1^2 & A_1^2 & B_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} B_1^2 & C_1^2 & A_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} \right) \\
&= \frac{1}{4} \begin{vmatrix} W_1^2 - A_1^2 & W_1^2 - B_1^2 & W_1^2 - C_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix},
\end{aligned}$$

where $W_1^2 = A_1^2 + B_1^2 + C_1^2$. This determinant is disassembled into two addend first of which is zero, since the top row are the same (W_1^2) which are proportional coefficients of the lower row. Remains:

$$\Delta_x(S_9) = -\frac{1}{4} \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}. \quad (29)$$

Similarly, we find:

$$\Delta_y(S_9) = +\frac{1}{4} \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix}. \quad (30)$$

Middle triangle sides opposite the vertex $A(A_x, A_y)$, $B(B_x, B_y)$ and $C(C_x, C_y)$ are points $A_1(A_{1x}, A_{1y})$, $B_1(B_{1x}, B_{1y})$ i $C_1(C_{1x}, C_{1y})$ given by formulas

(27). In these expressions the $A_1^2 = A_{1x}^2 + A_{1y}^2$, $B_1^2 = B_{1x}^2 + B_{1y}^2$ i $C_1^2 = C_{1x}^2 + C_{1y}^2$.

Thus we find coordinates of the nine-points center $S_9(S_{9x}, S_{9y})$ of the triangle ABC:

$$S_{9x} = - \frac{\begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}}, \quad (31)$$

$$S_{9y} = + \frac{\begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}}. \quad (32)$$

□

Example 5. Show that the center of the nine-point circle is middy-point of ortho-center and the center of the circumscribed circle of the triangle.

Solution. We are familiar with coordinates of the center of the nine-point S_9 , orthocenter H and the center of the circumscribed circle F, respectively:

$$2\mu = \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix},$$

$$S_{9x} = -\frac{1}{2\mu} \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}, \quad S_{9y} = +\frac{1}{2\mu} \begin{vmatrix} A_1^2 & B_1^2 & C_1^2 \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix}, \quad (33)$$

$$\begin{cases} H_x = -\frac{1}{2\mu} \begin{vmatrix} B_x C_x + B_y C_y & C_x A_x + C_y A_y & A_x B_x + A_y B_y \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} \\ H_y = +\frac{1}{2\mu} \begin{vmatrix} B_x C_x + B_y C_y & C_x A_x + C_y A_y & A_x B_x + A_y B_y \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix} \end{cases}, \quad (34)$$

$$\begin{cases} F_x = +\frac{1}{4\mu} \begin{vmatrix} A_x^2 + A_y^2 & B_x^2 + B_y^2 & C_x^2 + C_y^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} \\ F_y = -\frac{1}{4\mu} \begin{vmatrix} A_x^2 + A_y^2 & B_x^2 + B_y^2 & C_x^2 + C_y^2 \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix} \end{cases}.$$

Put $W_1^2 = A_1^2 + B_1^2 + C_1^2$ and the circumcenter:

$$\begin{aligned} F_x &= +\frac{1}{4\mu} \begin{vmatrix} W_1^2 - B_1^2 - C_1^2 & W_1^2 - C_1^2 - A_1^2 & W_1^2 - A_1^2 - B_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} = \\ &= +\frac{1}{4\mu} \begin{vmatrix} W_1^2 & W_1^2 & W_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} - \frac{1}{4\mu} \begin{vmatrix} B_1^2 + C_1^2 & C_1^2 + A_1^2 & A_1^2 + B_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}, \end{aligned}$$

A first determinant is zero, because it has proportional first and third rows. Similar work for the second coordinate and we find:

$$\begin{cases} F_x = -\frac{1}{4\mu} \begin{vmatrix} B_1^2 + C_1^2 & C_1^2 + A_1^2 & A_1^2 + B_1^2 \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix} \\ F_y = +\frac{1}{4\mu} \begin{vmatrix} B_1^2 + C_1^2 & C_1^2 + A_1^2 & A_1^2 + B_1^2 \\ A_x & B_x & C_x \\ 1 & 1 & 1 \end{vmatrix} \end{cases} \quad (35)$$

It also coordinates the center of the circumscribed circle of the given triangle. Now, it is easy to find the sum:

$$S_{9x} = \frac{H_x + F_x}{2}, \quad S_{9y} = \frac{H_y + F_y}{2}, \quad (36)$$

which is supposed to prove. $\square$

Example 6. Given the triangle $A(m, 0), B(n, 0), C(0, p)$. Prove that the point S_9 has coordinates, respectively:

$$S_9 \left(\frac{m+n}{4}, \frac{p^2 - mn}{4p} \right), \quad (37)$$

and the point S_9 lies on the Euler's line too.

For the given the triangle ABC , the center of gravity G , orthocenter H and circumcenter F are:

$$G \left(\frac{m+n}{3}, \frac{p}{3} \right), H \left(0, -\frac{mn}{p} \right), F \left(\frac{m+n}{2}, \frac{mn+p^2}{2p} \right) \quad (38)$$

and the three points lie on the Euler's line, as shown in [1]: (16).

Solution. Middle points of opposite sides of vertex A, B, C are respectively:

$$A_1 \left(\frac{n}{2}, \frac{p}{2} \right), B_1 \left(\frac{m}{2}, \frac{p}{2} \right), C_1 \left(\frac{m+n}{2}, 0 \right).$$

These are the vertices of the triangle with the determinants:

$$\Delta(A_1 B_1 C_1) = \begin{vmatrix} A_{1x} & B_{1x} & C_{1x} \\ A_{1y} & B_{1y} & C_{1y} \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \frac{n}{2} & \frac{m}{2} & \frac{m+n}{2} \\ \frac{p}{2} & \frac{p}{2} & 0 \\ 1 & 1 & 1 \end{vmatrix} = \frac{p(n-m)}{4},$$

$$\Delta_x(A_1 B_1 C_1) = +\frac{1}{2} \begin{vmatrix} A_{1x}^2 + A_{1y}^2 & B_{1x}^2 + B_{1y}^2 & C_{1x}^2 + C_{1y}^2 \\ A_{1y} & B_{1y} & C_{1y} \\ 1 & 1 & 1 \end{vmatrix} = \frac{p(n^2 - m^2)}{16},$$

$$\Delta_y(A_1 B_1 C_1) = -\frac{1}{2} \begin{vmatrix} A_{1x}^2 + A_{1y}^2 & B_{1x}^2 + B_{1y}^2 & C_{1x}^2 + C_{1y}^2 \\ A_{1x} & B_{1x} & C_{1x} \\ 1 & 1 & 1 \end{vmatrix} = \frac{(n-m)(p^2 - mn)}{16}.$$

Accordingly, the center of nine-points of the triangle $A(m, 0), B(n, 0), C(0, p)$ is specified (31).

Second, from (30) are, slopes k of the straight lines GH i GS_9 :

$$k(GH) = \frac{G_y - H_y}{G_x - H_x} = \frac{\frac{p}{3} + \frac{mn}{p}}{\frac{m+n}{3} - 0} = \frac{p^2 + 3mn}{p(m+n)},$$

$$k(GS_9) = \frac{G_y - S_{9y}}{G_x - S_{9x}} = \frac{\frac{p}{3} - \frac{p^2 - mn}{4p}}{\frac{m+n}{3} - \frac{m+n}{4}} = \frac{p^2 + 3mn}{p(m+n)}. \quad (39)$$

Slopes of the lines GH and GS_9 are equal. The both lines have the same point G and $GH \parallel GS_9$. Accordingly, the point S_9 is incident to Euler's line. $\square$

Example 7. For the given triangle $A(0,0), B(m,0), C(n,p)$, Figure 4, the center of gravity G, orthocenter H and circumcenter F are:

$$G\left(\frac{m+n}{3}, \frac{p}{3}\right), H\left(n, \frac{mn-n^2}{p}\right), F\left(\frac{m}{2}, \frac{n^2+p^2-mn}{2p}\right). \quad (40)$$

Show that these three are collinear points.

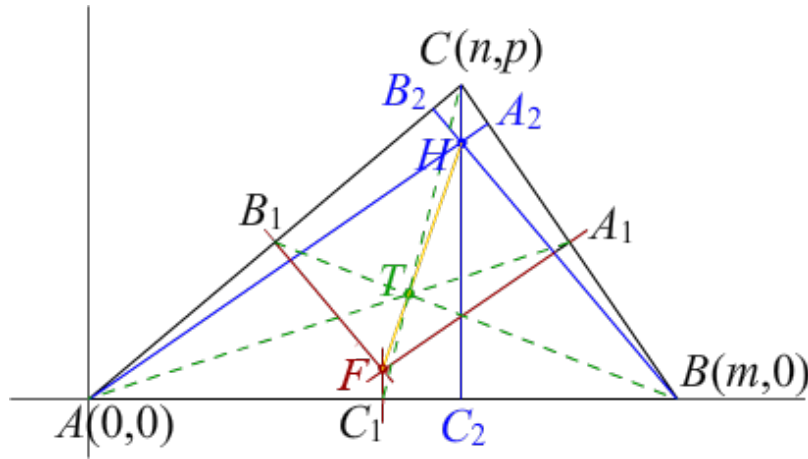


Figure 4: Triangle $A(0,0), B(m,0), C(n,p)$ and points T, H, F

Rešenje. Point G is trivial. Determinant of the systems (22-25) are:

$$\Delta(H) = \Delta(F) = \begin{vmatrix} 0 & m & n \\ 0 & 0 & p \\ 1 & 1 & 1 \end{vmatrix} = mp, \quad (41)$$

Which is common for orthocenter H and center F of circle passing through vertexes ABC. Determinants of coordinates of the orthocenter H:

$$\Delta_x(H) = - \begin{vmatrix} mn & 0 & 0 \\ 0 & 0 & p \\ 1 & 1 & 1 \end{vmatrix} = mnp, \quad \Delta_y(H) = \begin{vmatrix} mn & 0 & 0 \\ 0 & m & n \\ 1 & 1 & 1 \end{vmatrix} = mn(m-n).$$

Finally, the corresponding determinants of circumcenter F are:

$$\Delta_x(F) = + \frac{1}{2} \begin{vmatrix} 0 & m^2 & n^2 + p^2 \\ 0 & 0 & p \\ 1 & 1 & 1 \end{vmatrix} = \frac{m^2 p}{2},$$

$$\Delta_y(F) = - \frac{1}{2} \begin{vmatrix} 0 & m^2 & n^2 + p^2 \\ 0 & m & n \\ 1 & 1 & 1 \end{vmatrix} = \frac{-m^2 n + mn^2 + mp^2}{2}.$$

Finding quotients $\frac{\Delta_x}{\Delta}$ and $\frac{\Delta_y}{\Delta}$ confirm the accuracy of the (33).

Double the area's size of the triangle GHF, respectively:

$$\begin{aligned}
 2\mu &= \begin{vmatrix} \frac{m+n}{3} & n & \frac{m}{2} \\ \frac{p}{3} & \frac{mn-n^2}{p} & \frac{n^2+p^2-mn}{2p} \\ 1 & 1 & 1 \end{vmatrix} = \\
 &= \frac{m+n}{3} \begin{vmatrix} -\frac{m-2n}{3} & \frac{1}{2} \cdot \frac{m-2n}{3} \\ -\frac{3n^2+p^2-3mn}{3p} & \frac{1}{2} \cdot \frac{3n^2+p^2-3mn}{3p} \end{vmatrix} = \\
 &= \frac{m+n}{3} \cdot \frac{m-2n}{3} \cdot \frac{3n^2+p^2-3mn}{3p} \cdot \begin{vmatrix} -1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{vmatrix} = 0,
 \end{aligned}$$

which means that the center of gravity G, the orthocenter H and the circumcenter F of the triangle are the three collinear points. $\square$

Example 8. Given the triangle $A(0,0), B(m,0), C(n,p)$. Prove that the point S_9 has coordinates, respectively:

$$\left(\frac{m+2n}{4}, \frac{mn-n^2+p^2}{4p} \right), \quad (42)$$

and the point S_9 lies on the Euler's line, of course.

Solution. The middle points are:

$$A_1 \left(\frac{m+n}{2}, \frac{p}{2} \right), B_1 \left(\frac{n}{2}, \frac{p}{2} \right), C_1 \left(\frac{m}{2}, 0 \right).$$

Determinants are:

$$\Delta(A_1 B_1 C_1) = \begin{vmatrix} A_{1x} & B_{1x} & C_{1x} \\ A_{1y} & B_{1y} & C_{1y} \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \frac{m+n}{2} & \frac{n}{2} & \frac{m}{2} \\ \frac{p}{2} & \frac{p}{2} & 0 \\ 1 & 1 & 1 \end{vmatrix} = \frac{mp}{4},$$

$$\Delta_x(A_1 B_1 C_1) = +\frac{1}{2} \begin{vmatrix} A_{1x}^2 + A_{1y}^2 & B_{1x}^2 + B_{1y}^2 & C_{1x}^2 + C_{1y}^2 \\ A_{1y} & B_{1y} & C_{1y} \\ 1 & 1 & 1 \end{vmatrix} = \frac{mp(m+2n)}{16},$$

$$\Delta_y(A_1 B_1 C_1) = -\frac{1}{2} \begin{vmatrix} A_{1x}^2 + A_{1y}^2 & B_{1x}^2 + B_{1y}^2 & C_{1x}^2 + C_{1y}^2 \\ A_{1x} & B_{1x} & C_{1x} \\ 1 & 1 & 1 \end{vmatrix} = \frac{m(mn-n^2+p^2)}{16}.$$

Thus, the center of the nine-point of the triangle $A(0,0), B(m,0), C(n,p)$ is:

$$S_9 \left(\frac{m+2n}{4}, \frac{mn-n^2+p^2}{4p} \right). \quad (43)$$

Second, slopes k of the straight lines GH i GS_9 are:

$$\begin{aligned}
 k(GH) &= \frac{G_y - H_y}{G_x - H_x} = \frac{\frac{p}{3} - \frac{mn-n^2}{p}}{\frac{m+n}{3} - n} = \frac{-3mn + 3n^2 + p^2}{p(m-2n)}, \\
 k(GS_9) &= \frac{G_y - S_{9y}}{G_x - S_{9x}} = \frac{\frac{p}{3} - \frac{mn-n^2+p^2}{4p}}{\frac{m+n}{3} - \frac{m+2n}{4}} = \frac{-3mn + 3n^2 + p^2}{p(m-2n)}. \quad (44)
 \end{aligned}$$

Accordingly, the point S_9 is incident to Euler's line. $\square$

Note that any triangle by rotation and translation of the coordinate system can be placed at the position of the triangle of Examples 7 or 8, which is further evidence of general collinearity of points G, H and F. On the other hand, knowing that these points lie on a line, this is a partial confirmation of the accuracy of previous results.

6 TRILINEAR COORDINATES

Trilinear, or *trilinear coordinates* of point to a given triangle represent the relative distances of the three sides of the triangle. They are kind of *homogeneous coordinates*. They were discovered 1835 by German mathematician *Julius Plücker* (1801 – 1868) that gave large contributions to analytic geometry and physics.

Trilinear is regulated by a series of three numbers $\rho_a \rho_b \rho_c$ which is extended proportion, relations within r_a, R_B, r_c given point to the sides of the triangle a, b, c , as shown in Figure 1. That's why is written in the form:

$$\rho_a : \rho_b : \rho_c. \quad (45)$$

As is known, this is proportional distances, and is:

$$\rho_a : \rho_b : \rho_c = \lambda \rho_a : \lambda \rho_b : \lambda \rho_c, \quad (46)$$

for an arbitrary real nonzero number lambda ($\lambda \neq 0$). Exceptions are the vertices of a triangle A, B, C , which for simplicity is usually denoted respectively with:

$$1 : 0 : 0, \quad 0 : 1 : 0, \quad 0 : 0 : 1. \quad (47)$$

Significant points of the triangle are taken in order: center inscribed circle, center of gravity, circumcenter, ... and are marked with $X_1, X_2, X_3, \dots$, as it studied the American mathematician Clark Kimberling [3] since 1994.

A detailed list of more than five thousand Kimberling's points can be found in the Encyclopedia of Triangle Centers¹, and the first five are shown in Table 1. In the first column of the Kimberling tags dots here denoted respectively S, G, F, H and S_9 . In the second column are names of points, in the third are trilinears, the extended proportions like (45).

Table 1: Trilinears

No.	Point Name	Trilinear Coordinates
X_1	Incenter (S)	$1 : 1 : 1$
X_2	Centroid (G)	$\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$
X_3	Circumcenter (F)	$\cos \alpha : \cos \beta : \cos \gamma$
X_4	Ortocentar (H)	$\sec \alpha : \sec \beta : \sec \gamma$
X_5	Nine-point (S_9)	$\cos(\beta - \gamma) : \cos(\gamma - \alpha) : \cos(\alpha - \beta)$

In addition to the usual methods of mathematical literature, trilinear coordinates can be calculated directly from the Cartesian coordinates of the vertices' triangle. Starting from the general form of the line equation (16) and the expression (17) for the distance of point $P(P_x, P_y)$ to the side a, b and c of the triangle ABC , i.e. the system of equations:

$$\begin{cases} a_y P_x - a_x P_y - [B, C] = -ar_a, \\ b_y P_x - b_x P_y - [C, A] = -br_b, \\ c_y P_x - c_x P_y - [A, B] = -cr_c. \end{cases} \quad (48)$$

Then we form the extended proportion:

$$\lambda r_a : \lambda r_b : \lambda r_c, \quad (49)$$

taking arbitrary $\lambda \neq 0$.

¹ ETC: <http://faculty.evansville.edu/ck6/encyclopedia/ETC.html>

For the incenter of (X_1) we even do not need that much, because the point is equidistant from all three sides of the triangle, and the trilinear $1 : 1 : 1$ is obviously correct.

Example 9. Show that the trilinear $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$ is correct for centroid X_2 .

Solution. As we know, the coordinates of center of gravity (centroid) are:

$$G_x = \frac{A_x + B_x + C_x}{3}, \quad G_y = \frac{A_y + B_y + C_y}{3}.$$

and put them into the system (48). After rearrangement we find:

$$ar_a = \frac{2}{3}\mu, \quad br_b = \frac{2}{3}\mu, \quad cr_c = \frac{2}{3}\mu,$$

and hence the required proportion:

$$r_a : r_b : r_c = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

□

Theorem 5 (Sine Rule). When a, b, c are the lengths of sides of the triangle whose inner angles opposite to the given sides are, respectfully α, β, γ , and R is radius of circumcircle, than

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R. \quad (50)$$

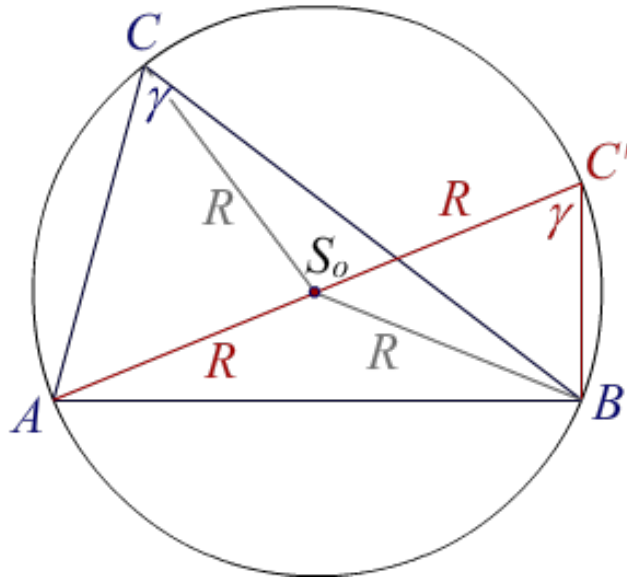


Figure 5: Sine Rule

Proof. In Figure 5 is given triangle. Choose an arbitrary vertex, say C . As the peripheral angles of the same chord are equal, it is $\angle C = \angle C'$. However, the angle $\angle C'BA$ is a right (90°), because it is the peripheral angle of diameter AC' . From the right triangle ABC' we have $\overline{AB} = \overline{AC'} \cdot \sin \gamma$, i.e. $\frac{a}{\sin \gamma} = 2R$. The other two equalities can be proved similarly. □

Example 10. Show that the trilinear $\cos \alpha : \cos \beta : \cos \gamma$ is correct for circumcenter X_3 .

Solution. In Figure 5, circle of radius R goes through vertices of the triangle A, B, C and the point C' . Peripheral angles of points C and C' over the same chord AB are equal to γ .

Triangle ABC' is rectangular, with the right angle vertex of B , the angle γ to the vertex C' and hypotenuse of $\overline{AC'} = 2R$. Hence the side $\overline{C'B} = 2R \cos \gamma$, and the height from the center of the circle S_o on side $C = AB$ is $r_c = R \cos \gamma$. Similarly we get $r_a = R \cos \alpha$ and $r_b = R \cos \beta$, then

$$r_a : r_b : r_c = \cos \alpha : \cos \beta : \cos \gamma,$$

which is intended to demonstrate. $\square$

In the fourth row of the table 1 is (secant angle) = $1 / (\cosine \text{ angle})$.

Trilinear coordinates can be normalized so that they represent the actual distance from a given point P to the appropriate sides. In Figure 1, distance $r_a = \lambda \rho_a$, $r_b = \lambda \rho_b$ i $r_c = \lambda \rho_c$ can be found by looking by area of triangles $\mu_a = \mu(PBC)$, $\mu_b = \mu(PCA)$ and $\mu_c = \mu(PAB)$. So we have:

$$\begin{aligned} \mu &= \mu_a + \mu_b + \mu_c = \\ &= \frac{1}{2} a r_a + \frac{1}{2} b r_b + \frac{1}{2} c r_c \\ &= \frac{1}{2} a \lambda \rho_a + \frac{1}{2} b \lambda \rho_b + \frac{1}{2} c \lambda \rho_c \\ &= \frac{\lambda}{2} (a \rho_a + b \rho_b + c \rho_c). \end{aligned}$$

Thus,

$$\lambda = \frac{2\mu}{a \rho_a + b \rho_b + c \rho_c}, \quad (51)$$

where a, b, c are sides, and μ is the area of the triangle ABC . When the coefficient $\lambda = 1$, then we have the actual distance to point P :

$$r_a : r_b : r_c. \quad (52)$$

Normalized trilinear coordinates are called *exact trilinear coordinates*, stated in the table 2.

Table 2: Exact Trilineares

Point name	Exact Trilinear
Incenter (S)	$r : r : r$
Centroid (G)	$\frac{bc}{6R} : \frac{ca}{6R} : \frac{ab}{6R}$
Circumcenter (F)	$R \cos \alpha : R \cos \beta : R \cos \gamma$
Ortocentar (H)	$2R \cos \beta \cos \gamma : 2R \cos \gamma \cos \alpha : 2R \cos \alpha \cos \beta$
Nine-point (S_9)	$\frac{R}{2} \cos(\beta - \gamma) : \frac{R}{2} \cos(\gamma - \alpha) : \frac{R}{2} \cos(\alpha - \beta)$

Specific point of the triangle would be those that exist under a general definition, regardless of the choice of the triangle. For example, the incenter of X_1 is a point equidistant from all three sides of the triangle. The second is centroid, a single point where intersects all three medians, and its distance to the sides inversely proportional to the length of the sides.

The following is the algebraic generalization of these invariants. This is the definition Kimberling's center of the triangle.

If the point P has trilinear representation:

$$f(A, B, C) : g(A, B, C) : h(A, B, C), \quad (53)$$

such that is valid

$$\begin{aligned} \text{i: } & (A, B, C) = f(B, C, A), \quad h(A, B, C) = f(C, A, B), \\ \text{ii: } & f(A, C, B) = f(A, B, C), \end{aligned} \quad (54)$$

then the representation does not change by replacing the vertices of the triangle. In addition, if a point P can be represented by using extended proportions

$$u(a, b, c) : u(b, c, a) : u(c, a, b), \quad (55)$$

where u is homogeneous function of the variables a, b and c, sides of a given triangle, then P is called *center of triangle*, or simply *center*.

Kimberling *main center* is a point whose trilinear representation (53) has the form:

$$f(A) : f(B) : f(C). \quad (56)$$

The first four points of the table 1 and Fermat point (X_{13}), whose trilinear coordinates (cosecant angle = 1 / sinus angle):

$$\csc(A + \frac{\pi}{3}) : \csc(B + \frac{\pi}{3}) : \csc(C + \frac{\pi}{3}) \quad (57)$$

are five main Kimberling centers listed in the paper [3], upon which were later defined classes of these centers.

7 PLANE IN SPACE

Given a 3D Euclidean space with Cartesian coordinate system Oxyz. Unit vector $\vec{n}$, which forms the angles ω_x , ω_y and ω_z with the coordinate axes is:

$$\vec{n} = \cos \omega_x \vec{i} + \cos \omega_y \vec{j} + \cos \omega_z \vec{k}. \quad (58)$$

If the point $P(p_x, p_y, p_z)$ is in the plane α perpendicular to the vector $\vec{n}$, p away from the origin O, then the scalar product of vectors $\vec{n}$ and $\vec{OP}$ is equal to the distance from the origin of the plane, so that:

$$p_x \cos \omega_x + p_y \cos \omega_y + p_z \cos \omega_z = p. \quad (59)$$

Naimely, the angle between $\vec{n}$ and $\vec{OP}$, say ϕ , so $\vec{n} \cdot \vec{OP} = P \cos \phi$. The Figure 6 shows that $P \cos \phi = p$. On the other hand, the dot product of these vectors (59) gives the so-called *normal form* plane equation. It is generalization of normal form of the equation of a straight line (16) from 2D to the space 3D.

The *segment formula* of the plane in space is:

$$\frac{x}{m_x} + \frac{y}{m_y} + \frac{z}{m_z} = 1 \quad (60)$$

where m_x , m_y and m_z are points (values) of abscissa, ordinate and applicate intersection with the given plane.

Consequently, *general form* of the plane equation (α) is:

$$\alpha_1 x + \alpha_2 y + \alpha_3 z = \alpha_0. \quad (61)$$

If the plane containing the points A, B and C with coordinates $Q(Q_x, Q_y, Q_z)$, for every $Q \in \{A, B, C\}$, then holds the system of equations:

$$\begin{cases} A: & \alpha_1 A_x + \alpha_2 A_y + \alpha_3 A_z = \alpha_0, \\ B: & \alpha_1 B_x + \alpha_2 B_y + \alpha_3 B_z = \alpha_0, \\ C: & \alpha_1 C_x + \alpha_2 C_y + \alpha_3 C_z = \alpha_0. \end{cases} \quad (62)$$

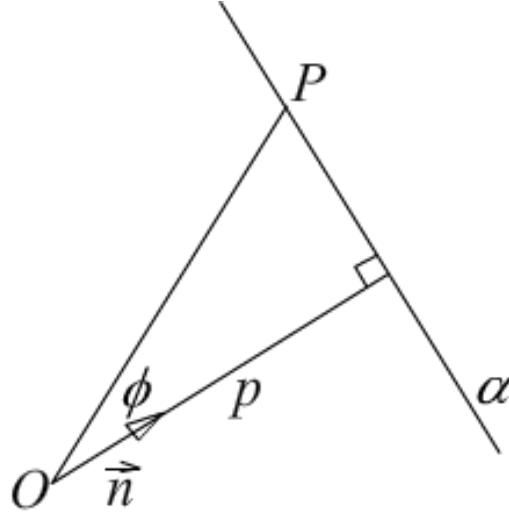


Figure 6: Normal $\vec{n}$ to the plane α and the point $P \in \alpha$

The determinant of this system is:

$$\Delta(ABC) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = V(ABC). \quad (63)$$

It is a mixed product of vectors $\vec{A} = \overrightarrow{OA}$, $\vec{B} = \overrightarrow{OB}$ and $\vec{C} = \overrightarrow{OC}$ whose value is the volume of prisms stretched with these three vectors. Prism base is a triangle ABC, the fourth vertex is origin O, so the height of the prism (see 6) is distance p of the normal form of the plane equation (59). Accordingly, the area of triangle ABC is:

$$\mu(ABC) = \frac{V(ABC)}{p}, \quad p = \frac{\alpha_0}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}, \quad (64)$$

where p is obtained by *normalization* of plane of equation (61), i.e. dividing the equation with $\alpha = \pm \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$ and by reduction to the form:

$$\frac{\alpha_1}{\alpha}x + \frac{\alpha_2}{\alpha}y + \frac{\alpha_3}{\alpha}z = \frac{\alpha_0}{\alpha}. \quad (65)$$

The sign of the number of alpha (context distinguished from the number of planes alfa) is chosen so that $p = \frac{\alpha_0}{\alpha} \geq 0$, i.e. the expression on the right side of equation (65) is nonnegative. Comparing (65) with (59) we see that the vector $\vec{n}$, which is perpendicular to the given plane built with the coordinate axes angles whose cosines:

$$\cos \omega_x = \frac{\alpha_1}{\alpha}, \quad \cos \omega_y = \frac{\alpha_2}{\alpha}, \quad \cos \omega_z = \frac{\alpha_3}{\alpha}. \quad (66)$$

Let's return to the system (62). Subtracting the first equation from the second, second from third, and third from the first equation we obtain the equivalent system:

$$\begin{cases} AB: & \alpha_1(B_x - A_x) + \alpha_2(B_y - A_y) + \alpha_3(B_z - A_z) = 0, \\ BC: & \alpha_1(C_x - B_x) + \alpha_2(C_y - B_y) + \alpha_3(C_z - B_z) = 0, \\ CA: & \alpha_1(A_x - C_x) + \alpha_2(A_y - C_y) + \alpha_3(A_z - C_z) = 0. \end{cases} \quad (67)$$

The sum of all three equations is the identity $0 = 0$. This means that the determinant of the system is zero and that the system is consistent. In addition to trivial $A = B = C$, there are other solutions.

Oriented projection of the vector $\vec{a} = \overrightarrow{BC}$, $\vec{b} = \overrightarrow{CA}$ i $\vec{c} = \overrightarrow{AB}$ to the axes are:

$$\begin{cases} a: & a_x = C_x - B_x, \quad a_y = C_y - B_y, \quad a_z = C_z - B_z, \\ b: & b_x = A_x - C_x, \quad b_y = A_y - C_y, \quad b_z = A_z - C_z, \\ c: & c_x = B_x - A_x, \quad c_y = B_y - A_y, \quad c_z = B_z - A_z. \end{cases} \quad (68)$$

Now the system (67) can be written in a shorter form:

$$\begin{cases} a: & \alpha_1 a_x + \alpha_2 a_y + \alpha_3 a_z = 0, \\ b: & \alpha_1 b_x + \alpha_2 b_y + \alpha_3 b_z = 0, \\ c: & \alpha_1 c_x + \alpha_2 c_y + \alpha_3 c_z = 0. \end{cases} \quad (69)$$

We know that the determinant of this system is zero, but its meaning is now the volume of prisms whose edges define the vectors of a given triangle sides:

$$\Delta(abc) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = V(abc), \quad (70)$$

The projections of the given triangle to the coordinate planes $x = 0$, $y = 0$ and $z = 0$ are triangles with commutators of the points:

$$\begin{aligned} [A, B]_x &= A_y B_z - A_z B_y, & [B, C]_x &= B_y C_z - B_z C_y, & \dots, \\ [A, B]_y &= A_z B_x - A_x B_z, & [B, C]_y &= B_z C_x - B_x C_z, & \dots, \\ [A, B]_z &= A_x B_y - A_y B_x, & [B, C]_z &= B_x C_y - B_y C_x & \dots \end{aligned} \quad (71)$$

Surfaces of projected triangles are:

$$\begin{cases} Oyz: & \mu_x = [A, B]_x + [B, C]_x + [C, A]_x, \\ Ozx: & \mu_y = [A, B]_y + [B, C]_y + [C, A]_y, \\ Oxy: & \mu_z = [A, B]_z + [B, C]_z + [C, A]_z. \end{cases} \quad (72)$$

Similarly, we define commutators of projections of sides (edges) of the triangles, projected to the coordinate planes $x = 0$, $y = 0$ i $z = 0$.

Note that in each projected plane, for example Oxy , that is $z = 0$ are true [2], the theorems about radii of enrolled circles of a triangle (Theorems 5 and 6). Thus, the volume of (70) can be written:

$$V(abc) = a_x [b, c]_x + a_y [b, c]_y + a_z [b, c]_z = 0,$$

which is easy to calculate when the determinant $\Delta(abc)$ develop by the first row. Note that the second cofactor $-[b, c]_y$. Next we have:

$$V(abc) = a_x \mu_x + a_y \mu_y + a_z \mu_z = 0. \quad (73)$$

When the determinant develops by the second, then the third row, it gives the similar results, and we have the system of equations:

$$\begin{cases} 1. & a_x \mu_x + a_y \mu_y + a_z \mu_z = 0, \\ 2. & b_x \mu_x + b_y \mu_y + b_z \mu_z = 0, \\ 3. & c_x \mu_x + c_y \mu_y + c_z \mu_z = 0. \end{cases} \quad (74)$$

Let us turn again to the system (62). We have seen (66) that by the normalization it can be reduced to an equivalent system:

$$\begin{cases} A: & A_x \cos \omega_x + A_y \cos \omega_y + A_z \cos \omega_z = p, \\ B: & B_x \cos \omega_x + B_y \cos \omega_y + B_z \cos \omega_z = p, \\ C: & C_x \cos \omega_x + C_y \cos \omega_y + C_z \cos \omega_z = p. \end{cases} \quad (75)$$

Let's find the cosine angle perpendicular to the given plane α in the function of the vertices of the triangle $A, B, C \in \alpha$.

Determinant of system (75) is the volume (70) of prism whose base is given triangle, and the fourth vertex is the origin O of coordinate system. So

$$\Delta = V(ABC). \quad (76)$$

Determinant of the first cosine is:

$$\Delta_x = \begin{vmatrix} p & A_y & A_z \\ p & B_y & B_z \\ p & C_y & C_z \end{vmatrix} = p\mu_x, \quad (77)$$

or, is equal to the distance (p) of the plane α from the origin, multiplied by the area of projection of the given triangle to the plane $x = 0$.

Determinant of the second cosine is:

$$\Delta_y = \begin{vmatrix} A_x & p & A_z \\ B_x & p & B_z \\ C_x & p & C_z \end{vmatrix} = p\mu_y. \quad (78)$$

Note that $B_x C_z - C_x B_z = -[B, C]_y$ and similarly for the other cofactors, and determinant is developing by column with coefficients p .

Determinant of the third cosine is:

$$\Delta_z = \begin{vmatrix} A_x & A_y & p \\ B_x & B_y & p \\ C_x & C_y & p \end{vmatrix} = p\mu_z. \quad (79)$$

Thus we have arrived at the following theorem.

Theorem 6. *The plane that contains not the origin is given by a triangle ABC with area μ . Surface projection of the given triangle on the coordinate planes $x = 0$, $y = 0$ and $z = 0$ were respectively μ_x , μ_y i μ_z . Then the equation of the plane that contains the vertices of a given triangle is as follows:*

$$\mu_x x + \mu_y y + \mu_z z = p\mu. \quad (80)$$

Proof. Distance of the plane with the triangle from the origin is p , the area of the triangle is μ , and the surface projection of the triangle on the plane $x = 0$, $y = 0$ i $z = 0$ were respectively μ_x , μ_y i μ_z .

Starting from the system (75) we find the determinants (77-79). Therefore, the required equation of the plane (65) with coefficients (66), becomes:

$$\frac{\Delta_x}{\Delta} x + \frac{\Delta_y}{\Delta} y + \frac{\Delta_z}{\Delta} z = p,$$

$$\Delta_x x + \Delta_y y + \Delta_z z = p\Delta,$$

$$p\mu_x x + p\mu_y y + p\mu_z z = p^2\mu,$$

and dividing by $p \neq 0$ we obtain the desired equation (80). $\square$

Neposredne posledice ove teoreme su razni izrazi koji definišu jednačinu ravni pomoću pomenutih površina trougla ABC sadržanog u toj ravni.

Corollary 1. *The area of the triangle $\mu(ABC)$ and the areas of the projections of the triangle Π_x , Π_y , Π_z to the coordinate planes $x = 0$, $y = 0$, $z = 0$ are in relations:*

$$\mu_x = \mu \cos \omega_x, \quad \mu_y = \mu \cos \omega_y, \quad \mu_z = \mu \cos \omega_z, \quad (81)$$

when the plane of the triangle contains not the origin.

Proof. From the relation (80) we obtain:

$$\frac{\mu_x}{\mu}x + \frac{\mu_y}{\mu}y + \frac{\mu_z}{\mu}z = p.$$

By comparing with the normal form of the plane equation (59) follow the required relation (81). $\square$

Example 11. Show that the projection c_x, c_y of line AB and commutator points in the plane Oxy are in relation:

$$c_y = c \cos \omega, \quad c_x = -c \sin \omega, \quad [A, B] = pc, \quad (82)$$

where p is the distance from origin to line defined by points $A(A_x, A_y)$ and $B(B_x, B_y)$, and c is length of AB.

Solution. The general form of the line equation is $c_y x - c_x y = [A, B]$. When this equation divided by the $c = \overline{AB}$ we obtain:

$$\frac{c_y}{c}x - \frac{c_x}{c}y = \frac{[A, B]}{c}.$$

When this result is compared with the normal form of the equation in the plane we get:

$$\frac{c_y}{c} = \cos \omega, \quad -\frac{c_x}{c} = \sin \omega, \quad \frac{[A, B]}{c} = p,$$

and hence the required relation. $\square$

Comparing the segmental form (60) of the line equation with theorem 6 find the relations:

$$m_x = \frac{p\mu}{\mu_x}, \quad m_y = \frac{p\mu}{\mu_y}, \quad m_z = \frac{p\mu}{\mu_z}, \quad (83)$$

where p distance plane α from the origin, and m_x, m_y i m_z abscissa, ordinate and applicate points of intersection plane and the coordinate axes. Area of the triangle $ABC \subset \alpha$ is Π , and the projection of the triangle on the coordinate plane have areas μ_x, μ_y i μ_z . Note that $p\mu = V$, i.e. volume of a prism whose base is the triangle and the fourth vertex is the origin O of coordinate system.

Example 12. Apply these formulas to commutators (5) defined by Figure 1.

Solution. Instead the plane α in space, we have the line in Oxy, so:

$$[A, B] = cd_c, \quad [B, C] = bd_a, \quad [C, A] = bd_b, \quad (84)$$

where d_i is length of the projection of the point O to the sides of the triangle, respectively $p \in \{a, b, c\}$. $\square$

Of course, $[A, B] = cd_c$ is double of the oriented area of the triangle ABO, etc. and the sum of all three of these areas is double the area of the triangle ABC.

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